Full description of the project **Invariants and methods of applied topology**

Main topics of the project

1. Variants of the Bourgin–Yang version of the Borsuk–Ulam theorem and their applications to combinatorics and equipartition problems.

2. A study of topological complexity of spaces with the presence of actions of finite groups.

3. A construction of linearization and spectrum of homomorphism of residually nilpotent group.

4. Geometrically defined linear spaces used to obtain geometrically distinct solutions of variational problems with $O(N)$-symmetry.

Research project objectives

The general aim of the project is to develop theory of certain topological invariants used in the field of applied topology and studied in the previous work of investigators, though completely new directions are also included. The central thematic stream is concentrated in the two first topics, where we intend to use the experience of the principal investigator gained through his past work on the Borsuk–Ulam theorem and the Lusternik–Schnirelmann category theory. In particular, we aim to improve and generalize recent versions of the Bourgin–Yang theorem (a variant of the celebrated Borsuk–Ulam theorem) and, in effect, obtain new applications to combinatorics and equipartition problems. Continuation of investigation of the notion of topological complexity of spaces with a given group of symmetries consitutes the other direction in this part.

Another, more tentative, subject is the construction and examination of properties of notions of "linearization" and "spectrum" of a homomorphism of a finitely generated residually nilpotent group. Such a notion was already defined and studied for homomorphisms of torsion free nilpotent groups as it appears naturally in the study of maps between nilmanifolds. It would encode dynamical properties of a homomorphism, e.g. properties of a map of an aspherical space.

Finally, we plan to obtain a detailed description of functional spaces which are determined by subgroups of $O(N)$ and give series of geometrically distinct solutions of nonlinear variational problems with $O(N)$-symmetry.

A more detailed description of specific objectives follows.

1. Variants of the Bourgin–Yang version of the Borsuk–Ulam theorem and their applications to combinatorics and equipartition problems. The aims of this part of the project are:

i) The classical version of the Borsuk–Ulam theorem states that there does not exist a continuous map $f\colon S(\mathbb{R}^n) \to S(\mathbb{R}^m)$ of spheres which satisfies $f(-x) = -f(x)$, provided that $n > m$. On the other hand, the Bourgin–Yang theorem (also in its classical statement) says that if $f\colon S(\mathbb{R}^n) \to \mathbb{R}^m$ is a map satisfying $f(-x) = -f(x)$, then $\dim f^{-1}(0) \geq n - m - 1$. In particular, if $n > m$ then the dimension of this set is ≥ 0 , thus it is nonempty, and the conclusion of the Borsuk–Uam theorem follows.

There are numerous and far reaching generalizations of the Borsuk–Ulam theorem. On the other hand, there are not nearly as many papers on the Bourgin–Yang theorem. Furthermore, to the best of our knowledge, those which exist rely on having a sphere as the (co)domain of the equivariant map in question ([Mar-Mat-San1], [Mar-Mat-San2]) or treat a very specific situation ([Mun3], [Vol2], [Vol3]).

Our first objective is to complete the details of a general theorem of the Bourgin–Yang type for groups $G = \mathbb{Z}_{p^k}$, $(\mathbb{Z}_p)^k$, $\mathbb{S}^k$. We already have a thorough discussion and a preliminary version of results available ([Bl-Mar-Si1]), though certain estimates (especially when $G = (\mathbb{Z}_p)^k$) are still far from satisfactory. Our formulation of this theorem is analogous to the most general version of the Borsuk–Ulam theorem due to M. Clapp and D. Puppe [Cl-Pu2]. More precisely, we merely assume that X is compact and $(n-1)$-connected (or, even more generally, $(n-1)$-acyclic with respect to a properly chosen field of coefficients) and Y is of finite dimension, say m (respectively, of cohomological dimension m). Then, given a G-equivariant map $f\colon X \to Y$ and a closed G-invariant subspace $A \subseteq Y$, we estimate the dimension of the set $f^{-1}(A)$. Our result states that, roughly speaking, $\dim f^{-1}(A) \geq \phi(n) - \psi(m)$, where ϕ and ψ are certain monotonic functions depending on G and the structure of actions on X and Y. This readily extends the results of [Mar-Mat-San1] and [Mar-Mat-San2]. We believe, however, that both ϕ and ψ we use right now are suboptimal and can be chosen so that the obtained estimates are significantly better.

One major technical point regarding this objective which is worth pointing out is the use of equivariant K-theory K_G^* as a tool. This enables us to carry out finer calculations compared to classically used Borel cohomology theory. On the other hand, there were some related significant obstacles, since algebraic gradation in K_G^*-theory is not naturally related to gradation coming from filtration by skeletons, and also K_G^*-theory is functorial only with respect to proper maps. These problems have been overcame in [Mar-Mat-San1] and [Bl-Mar-Si1], respectively. Interesingly enough, if we specify our work for $G = \mathbb{Z}_{p^k}$ to the situation considered by Munkholm [Mun3], who used singular homology and obstruction theory as main tools, the results only partially overlap, and not at all in an obvious manner (both in range and numeric values). A minor goal is to make a detailed comparison with Munkholm's theorems. A similar comparison with Volovikov's results [Vol2] for $G = (\mathbb{Z}_p)^k$ is also in order.

ii) If $G = (\mathbb{Z}_p)^k$ or $G = \mathbb{T}^k$, the Borsuk–Ulam theorem says that the necessary condition for the existence of a G-map $S(V) \to S(W)$, where V and W are orthogonal G-representations such that $V^G = W^G = \{0\}$, is $\dim V^H \leq \dim W^H$ for any closed subgroup $H \subseteq G$ ([Mar1], [Cl-Pu2]). If $G = \mathbb{Z}_{p^k}$, this condition assumes the form $\frac{\dim V^H}{p^{k-1}} \leq \dim W^H$ ([Bar1]). (For $p > 2$ we take "dim" to mean the complex dimension). It is straightforward to prove that this condition is sufficient for $G = (\mathbb{Z}_p)^k$, ([Mar3]), and it was recently shown that this is also the case for $G = \mathbb{T}^k$ ([Bl-Mar-Si2]). The related result for $G = \mathbb{Z}_{p^k}$ constitutes the task (1.ii) of this project and appears to be essentially more difficult.

iii) This task is closely related to the previous one. We plan to tackle an analogous problem for an arbitrary finite cyclic group $G = \mathbb{Z}_k$. In this case the necessary condition of divisibility of Euler classes $\mathbf{e}(V) \mid \mathbf{e}(W)$ in cohomology $H^*(BG;\mathbb{Z})$, provided that $\mathbf{e}(V) \neq 0$ ([Mar3]), should be strengthened in order to be sufficient. There are some similar or partial results in this direction. If $G = (\mathbb{Z}_p)^k$ and EG^N denotes the N-th skeleton of the classifying space EG of G, then the Euler class $\mathbf{e}(W) \in H^*(BG;\mathbb{Z})$ is the unique obstruction for the existence of a G-equivariant map $EG^N \to S(W)$ [Sar]. Similar thing happens if V is a free G-representation of a cyclic group $G = \mathbb{Z}_k$ and k either an odd number or is of the form $2l$, where l is odd [Izy-Mar].

iv) If follows from Carlson's proof of the Segal conjecture $\hat{\mathcal{A}}(G) \cong \pi_G^*(BG)$ and from Laitinen's theorem on p-adic completion of the Burnside ring of a p-group G that if V is an infinite-dimensional G-representation and W is a finite-dimensional G-representation with $W^G = \{0\}$, then a G-equivariant map $S(V) \to S(W)$ cannot exist ([Bar2]). However, for an arbitrary p-group G there is no known effective information on how big the difference $\dim W - \dim V$ can be. We will attempt to find an integer n_0 (depending on G and W) with the property that if $\dim V > n_0$, then a G-equivariant map $S(V) \to S(W)$ does not exist. Restating what was mentioned in 1.ii) in this language yields $n_0 = \dim W$ if $G = (\mathbb{Z}_p)^k$, and $n_0 = \frac{\dim W}{p^{k-1}}$ if $G = \mathbb{Z}_{p^k}$. Our working conjecture is as follows: $n_0 = \min \frac{\dim W}{|H|}$, where and $H \subsetneq G$ a cyclic subgroup and $|H|$ denotes the order of H.

v) While the Borsuk–Ulam theorem ensures the existence of a solution of equipartition problems (both in the classical combinatorial setting [Mat], as well as in other theories [Bl-Zie], [Do-Ka], [Sab]), the Bourgin–Yang theorem gives qualitative information about the size of the set of all possible solutions. Up until now only a very special variant of the Bougin–Yang theorem (with $G = (\mathbb{Z}_p)^k$, V a free G-representation and W a regular one, albeit with "removed" fixed points) was used in combinatorial applications. It was studied for $k = 1$ in [Mun3] and finally proved in [Vol2]. An interpretation of our versions of the Bourgin–Yang theorem in the context of equipartition problems is another (admittedly, less precisely) posed objective of the project.

vi) The last topic of this thematic group is the completion of classification of all compact Lie groups with the "absolute" Borsuk–Ulam property: if, given a pair V, W of orthogonal G-representations such that $V^G = W^G = \{0\}$, there exists a G-map $S(V) \to S(W)$, then $\dim V \leq \dim W$. If we additionally assume that $W \subseteq V$ is a sub-representation, then every (compact Lie) group G with such a property is of the form of extension $e \to \mathbb{T} \subset G \to \Gamma \to e$, where $\mathbb{T}$ is a torus and Γ is a p-group. The converse also holds [Bar23]. Without this additional assumption, every group G with this property is of the form $e \to \mathbb{T} \subset G \to \Gamma \to e$, where the order of each element of Γ is equal to p. On the other hand, any group which is the extension $e \to \mathbb{T} \subset G \to (\mathbb{Z}_p)^k \to e$ has this stronger Borsuk–Ulam property ([Mar2]). We aim to fill in this gap by classifying groups having the second Borsuk–Ulam property. Our conjecture is as follows: this is the class of extensions of the form $e \to \mathbb{T} \subset G \to (\mathbb{Z}_p)^k \to e$.

2. A study of topological complexity of spaces with the presence of actions of finite groups. This thematic group consists of:

i) In order to measure complexity of the process of motion planning of a mechanical system (a robot), M. Farber [Fa1], [Fa2], [Fa3] introduced the notion of topological complexity, $\mathrm{TC}(X)$, of a topological space X. This is defined as the minimal number of domains of continuity whose union covers

$X \times X$, where "domain of continuity" is taken to mean an open subset $U \subseteq X \times X$ such that a motion planning algorithm exists over U (i.e. there exists a continuous function $s\colon U \to PX$ such that $s(x,y)(0) = x$ and $s(x,y)(1) = y$ for any pair $(x,y) \in U$). Topologically, this corresponds to the Švarc category of the path fibration $PX \to X \times X$. Due to its applications in topological robotics — its knowledge is of practical use when designing optimal motion planners — and close relation to Lusternik–Schnirelmann category, topological complexity has attracted plenty of attention in recent years.

Mechanical systems often come equipped with symmetries visible in their configuration spaces, thus it not surprising that there have been attempts at weaving symmetries into the definition of topological complexity. We propose to study properties of these "symmetric" invariants. In fact, there are at least four different approaches, depending on how one decides to interpret the additional structure. One can rigidify planners ([Co-Gr]), locally simplify them at the cost of increasing the global amount of domains of continuity ([Lu-Mar]), simplify motion planners globally ([Bł-Kal2]), or "stabilize" them by twisting via $X \times_G EG$ ([Dr1]).

Our studies will include computation of these invariants for various G-spaces ([Bl-Lü-Zi], [Bł-Kal1], [Bł-Kal2], [Dr1], [Fr-Pa], [Go-Gra-To-Xi]), as well as relating them to classical invariants of transformation group theory ([Bł-Kal1]). We will also consider modifying the existing definitions to determine an invariant which would be effectively computable and the most adequate from the point of view of applications.

ii) The second objective in this group constitutes a more specific task: we intend to determine a construction of the G-Reeb graph $\mathcal{R}_G(f)$ [Re], [Ma-Sae],[Sha], i.e. a graph associated with a smooth G-invariant function on a G-manifold X. Originally, the Reeb graph was defined as a quotient $X/{\sim_{\mathcal{R}}}$, where $x \sim_{\mathcal{R}} y$ if $f(x) = f(y) = c$ and x, y are in the same connected component of $f^{-1}(c)$, and typically is represented as a graph embedded into X. The Reeb graph describes the relation between topology of X and the orbits of a gradient field which joins critical points of f. This notion is extensively used in visualization problems (cf. [Bi-Gi-Sp-Fa]). In a recent paper [Kal-Mar-Si], investigators of the project introduced (in the general case) a construction of this graph as a subcomplex of X. This led to properties known to hold before for Morse functions only ([Co-Ed-Ha-Na]).

We plan to adapt the construction to the situation when an action of a finite group G on X is given and f is a G-invariant function. We expect to obtain a graph with an action of G which permutes vertices and edges. It is worth pointing out that a direct adaptation of the original definition of the Reeb graph results in an object with a very poor action of G (in fact, very often trivial), and so it preserves no interesting information.

3. A construction of linearization and spectrum of homomorphism of residually nilpotent group.

i)-ii) Here we aim to introduce the notion of linearization of a homomorphism $\phi\colon G \to G$ of a finitely generated residually nilpotent group G and, later on, describe its properties. The main idea is to use the sequence $A_n(\phi)$ of homomorphisms induced by ϕ on factors of the lower central tower.

The idea of such a construction arose from the study of linearization of homomorphisms of finitely generated torsion free nilpotent groups, which appears in the geometric setting of periodic points (see [Je-Mar] for more references) or, correspondingly, entropy ([Mar-Przl]) of a map $X \to X$ on a compact nilmanifold X. Purely formally, this construction can be carried out for any group, but it has interesting (or required) properties only for residually nilpotent groups. At the moment this task is posed in general terms only, but it should be thought of as an attempt at studying group homomorphisms by means of geometric methods.

4. Geometrically defined linear spaces used to obtain geometrically distinct solutions of variational problems with $O(N)$-symmetry. We plan to show that a recently established theorem of the principal investigator [Mar4] on a construction and number of mutually linearly independent subspaces of functional spaces of functions on $\mathbb{R}^N$ (or D^N) can be applied to various variational problems with $O(N)$-symmetry. Moreover, a certain principle of composition of these subspaces can be used to find subspaces consisting of functions with a large set of zeroes (the nodal set). The mentioned theorem says that with every partition $\pi(N)$ of a natural number N into summands with a non-trivial Weyl group $W\big(\pi(N)\big)$ (this means that there are at least two summands which are equal) we can associate a functional subspace which consists functions changing sign and whose zero sets contain a union of hyperplanes of fixed points of reflections (transpositions) of $W(\pi(N))$. By specifying families of special partitions and using a refined version of an argument from [Kr-Mar], we have shown that the number of mutually orthogonal subspaces of this type has the rate of growth in N not smaller than $P\big([N/4]\big) \sim \frac{1}{N\sqrt{3}}\, e^{\pi\sqrt{N/6}}$, where $P(N)$ denotes the number of possible partitions of N (i.e. the number of ways of writing N as a sum of positive integers, where the order of addends is not significant). In effect, it is now known this rate of growth is exponential, which improves the previous logarithmic $\left[\log_2 \frac{N+2}{3}\right]$ ([Ba-Wil]) and linear $\left[\frac{N-3}{2}\right] + (-1)^N$ ([Kr-Mar]) rates.

Another direction of research is the corresponding study of vector-valued functions (in, say, $\mathbb{R}^d$). In order to define this kind of subspaces one ought to use $\tilde{d} \mid d$-dimensional irreducible representations of the group $W(\pi(N)) \subset \Sigma(N)$ of permutations. It is worth pointing out that in the case when we do not want to obtain many mutually linearly independent subspaces, there is no do need to increase N, but make use of finite subgroups of $O(N)$ generated by reflections instead, e.g. the Coxeter groups.

<u>Significance of the project</u>

1. Variants of the Bourgin–Yang version of the Borsuk–Ulam theorem and their applications to combinatorics and equipartition problems. The classical Borsuk–Ulam theorem for maps preserving $\mathbb{Z}_2$-symmetry has more then five hundreds generalizations and specifications (cf. [Ste] for a review of literature). The problem of understanding how large the class of symmetries for which it still holds has attracted an attention of several authors (cf. [Bar23] [Mar2] for more references). There are two main reasons behind this substantial amount of interest: applications to questions on multiplicities of solutions of variational problems with symmetry (e.g. the Ambrosetti–Rabinowitz symmetric mountain pass theorem; see [Bar2] for a survey of several results) on one hand, and to problems in combinatorics (e.g. the topological Tveberg theorem [Tve] by Barany, Shlosman, and Szücs [Sar], [Ziv1], the solution of the necklace problem by Alon and West, the answer to the Kneser conjecture on chromatic number of the Kneser graph by Lovasz; see [Mat] for further examples) on the other. It has also numerous other applications – let us only mention game theory [Sch-Si-Sp-To].

Our aim, however, is to study variants of the Bourgin–Yang theorem and its applications. Up until now it has been applied neither to combinatorial nor nonlinear problems with symmetry. Especially the former seems to be potentially very interesting: roughly speaking, if the Borsuk–Ulam theorem implies a solution of a given problem, then the Bourgin–Yang theorem estimates the size of the set of solutions. We have already proved the Bourgin–Yang theorem for distinct symmetries (groups), and so we expect that applications cover many cases.

The following version of the Bourgin–Yang problem has been studied extensively by a few mathematicians. Let $G = \{1, g_1, \dots, g_r\}$ be a finite group, X a G-space, Y any space (not necessarily equipped with a G-action!) and $f\colon X \to Y$ a map. The task at hand is to estimate the dimension of the set $A_f := \{x \in X \mid f(x) = f(g_1 x) = \cdots = f(g_r x)\}$ from below. This can be tackled by a clever use of the Bourgin–Yang theorem. Most significant results in this direction are due to H. Munkholm [Mun1], [Mun3] and A. Volovikov [Vol2], [Vol3], [Vol4]. The first author studied the case $G = \mathbb{Z}_{p^k}$, p an odd prime, $X = S^{2n-1}$ equipped with a a free action G-action, and $Y = \mathbb{R}^m$. The second author proved a related theorem for the p-torus $G = (\mathbb{Z}_p)^k$. The latter was applied by several authors to problems that arose from combinatorics ([Vre-Ziv1], [Vre-Ziv3], [Ziv1]), but only with the Borsuk–Ulam conclusion as the input. Any application which actually uses the full conclusion of the Bourgin–Yang theorem would be of importance.

All other tasks of this thematic group (1.ii-iv and 1.vi) are classical questions of equivariant topology. Some of them, e.g. (1.iii), are considered to be hard problems, so that any progress in solving them would be of importance.

2. A study of topological complexity of spaces with the presence of actions of finite groups. As already mentioned, the theory of topological complexity is one of youngest — it was introduced at the beginning of this century — and most rapidly developing areas of applied topology. Its branch devoted to the study of "symmetric" topological robotics is even more recent – the first paper on the topic, [Co-Gr], appeared only in 2012. Yet there already are hints that it will gain a significant impetus. Similarly as ordinary topological complexity is closely related to classical problems in algebraic topology (e.g. TC of the real projective space $\mathbb{R}P^n$ is well-known to be equal to the immersion dimension of $\mathbb{R}P^n$), equivariant topological complexity theories provide tools for studying classical topics in transformation group theory. For example, invariant complexity introduced in [Lu-Mar], combined with earlier results on the Lusternik–Schnirelmann category theory [Dr-Ka-Ru], [Gom-La-Acu], provides a criterion for a smooth action of $\mathbb{Z}_p$ on a sphere to be equivalent with a linear action [Bł-Kal1].

The bottom line is that we strongly believe that it is important for the state of mathematical research in Poland to be essentially involved in this research area.

The construction of a G-equivariant version of the Reeb graph of a G-invariant smooth function $X \to X$ is of interest because of the study a complexity of a G-action on X, as well as for improved visualization purposes.

3. A construction of linearization and spectrum of homomorphism of residually nilpotent group. The importance of the third topic of the project is in its novel approach to the study of group homomorphisms. Up until now, in geometric group theory more attention has been given to groups rather than their homomorphisms, in spite of the fact that the celebrated work of M. Gromov [Gro] which originated the theory was motivated by a problem that came from the study of homomorphisms. The notion of the rate of growth of a homomorphism (the logarithmic word growth, cf. [Je-Mar] [Mar-Prz2] for references) is used in many places, but estimates obtained by it are not optimal in certain geometric applications. To get a finer, in fact the best in general, estimate of the asymptotic Nielsen number $N_\infty(f)$ or, correspondingly, of topological entropy $\mathbf{h}(f)$, of a map $f\colon X \to X$ of a nilmanifold X, the authors of [JBLe-KBLe], [Je-Mar], [Mar-Prz1] and [Mar-Prz2] used the spectral radius of the full exterior power $\wedge^* A(\phi)$ of the homomorphism $\phi = \pi_1(f)\colon \pi_1(X) \to \pi_1(X)$ induced by f on the fundamental group of X, and $A(\phi)$ a linear map of the linearization of ϕ. The latter can be defined in purely group-theoretic terms by means of the lower central tower of group. This suggests that its counterpart defined in a more general case would carry some essential information about ϕ.

4. Geometrically defined linear spaces used to obtain geometrically distinct solutions of variational problems with $O(N)$-symmetry. The last objective is noteworthy not only as an essential improvement of earlier results, but also for its significant potential as a new technique. The first paper on the topic has recently been completed and the resulting progress is briefly described in the "Research objectives" section above. We plan to apply it to other variational problems with $O(N)$-symmetry to show capability of the method. Furthermore, the use of spaces of ρ-interwinding functions [BCM], [Mar4] (i.e. functions twisted by an irreducible representation of a subgroup of the symmetry group of the domain) can potentially be applied to show the existence and multiplicity of solutions of PDE variational problems with large (controlled) nodal set.

Work plan

A substantial part of habitual preparation of research has already been done. Some of the objectives have been studied in a very recent work of investigators. The results, however, are in varying degrees of completion. We intend to intensify the "classical way" of mathematical work: looking over the literature, discussing, presenting results during seminars, taking part in meetings, visiting and hosting domestic and foreign collaborators, and finally completing the results and preparing them for publication. We plan to publish several papers as an effect of our study. In the meantime we will present obtained results during conferences, workshops and invited lectures.

None of the listed topics should have special terms for their realization — we assume parallel work on all of them. This means that the project does not need a detailed timetable.

Research methodology

Below we list a selection of tools, techniques and theories we will use in order to achieve particular objectives of this project.

1, i–ii) Equivariant obstruction theory, K_G^*-theory, Borel cohomology $H_G^*(-;\mathbb{Z}_p)$, equivariant cohomological length with respect to a family of subgroups, direct constructions of G-maps. Localization theorems for Borel cohomology.

1, iii) Obstruction theory, calculus of Euler classes of vector bundles associated with V and W in various cohomology theories, e.g. in classical singular cohomology and those mentioned above.

1, iv) Atiyah–Segal completion and localization theorems in K_G^*-theory.

1, v) A methodology is not yet decided in detail.

1, vi) Obstruction theory, direct constructions, theory of representations of finite groups.

2, i) Equivariant homotopy and homology theories, equivariant Lusternik–Schnirelmann category, theorems on description and classification of finite group actions on manifolds.

2, ii) Direct construction, known characterizations of invariant functions (Morse functions), properties of equivariant flows, description of actions of finite groups on surfaces.

3, i) Definition of linearization of homomorphism ϕ by use of the sequence $A_n(\phi)$ of homomorphisms induced by ϕ on the sequence of factors of lower central tower. Elements of the geometric groups theory and general group theory

3, ii) The rate of growth of group, the rate of growth of homomorphism, generating sequence of power and exponential series.

4 Linear algebra, certain special subgroups of the orthogonal group $O(N)$, combinatorics of the theory of partitions of number N into summands, representation theory, variational principles of showing the existence of critical points by means of the mountain and symmetric mountain pass theorem.

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Part 1.

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